Comment on "Fluctuations of elastic interfaces in fluids: Theory, lattice-Boltzmann model, and simulation"

S. V. Lishchuk and C. M. Care
Materials and Engineering Research Institute, Sheffield Hallam University, Howard Street, Sheffield S1 1WB, United Kingdom
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The formulas for the force exerted by the interface upon the fluids, given by Stelitano and Rothman [Phys. Rev. E **62**, 6667 (2000)] are corrected.

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In Ref. [1] the expression for the force exerted by the interface upon the fluids was obtained by applying Hamilton's principle to the following Lagrangian describing the interface dynamics:

$$\mathcal{L}_{id} = \int\int \left(\frac{\rho}{2}\dot{\mathbf{r}}^2 - \frac{\epsilon}{2}H^2 - \sigma\right) dS dt. \tag{1}$$

Here ρ is the density of the hypersurface, ϵ is the bending rigidity, H is the mean curvature, σ is the surface tension, dS is the volume element for the parametrization $\mathbf{r} = \mathbf{r}(u_1, \ldots, u_n)$ of the n-dimensional hypersurface embedded in the $(n+1)$-dimensional space, $\dot{\mathbf{r}} = \partial\mathbf{r}/\partial t$ is the velocity. The parametrization was assumed to satisfy the condition

$$\frac{\partial \mathbf{r}}{\partial u_i} \cdot \frac{\partial \mathbf{r}}{\partial u_j} = \delta_{ij}, \tag{2}$$

δ_{ij} being the Kronecker delta, which was used throughout the calculations. Generally, Eq. (2) is not valid after applying the variation $\delta\mathbf{r}$ to the locus $\mathbf{r}$ of the interface, but this was not taken into account in the derivation.

In the present comment, for simplicity, we demonstrate the derivation of the correct expression for the force corresponding to the Lagrangian (1) using the example of a one-dimensional interface embedded in the two-dimensional space. In this case the locus of the interface $\mathbf{r}$ depends on a single spatial parameter u. We shall assume that the spatial and time dependence $\mathbf{r} \equiv \mathbf{r}(u,t)$ is sufficiently well behaved. The metric tensor has a single component

$$g \equiv g(u,t) = \left(\frac{\partial \mathbf{r}}{\partial u}\right)^2. \tag{3}$$

The unit tangent vector $\mathbf{e} = g^{-1/2}(\partial\mathbf{r}/\partial u)$ and the unit normal vector $\mathbf{n}$ satisfy the relations $\partial\mathbf{n}/\partial u = \sqrt{g}H\mathbf{e}$ and $\partial\mathbf{e}/\partial u = -\sqrt{g}H\mathbf{n}$, and one can write

$$\frac{\partial^2 \mathbf{r}}{\partial u^2} = \frac{\partial}{\partial u}(\sqrt{g}\mathbf{e}) = -gH\mathbf{n} + \frac{1}{2\sqrt{g}}\frac{\partial g}{\partial u}\mathbf{e}. \tag{4}$$

It follows from (4) that

$$H^2 = \frac{1}{g^2}\left(\frac{\partial^2 \mathbf{r}}{\partial u^2}\right)^2 - \frac{1}{4g^3}\left(\frac{\partial g}{\partial u}\right)^2. \tag{5}$$

Casting the volume element as $dS = \sqrt{g}du$ and using (3) and (5) we can recast the Lagrangian (1) as

$$\mathcal{L}_{id} = \int\int L_{id} du dt, \tag{6}$$

where

$$L_{id} = \frac{\rho}{2}\dot{\mathbf{r}}^2 - \left\{\frac{\epsilon}{2g^2}\left[\left(\frac{\partial^2 \mathbf{r}}{\partial u^2}\right)^2 - \frac{1}{g}\left(\frac{\partial \mathbf{r}}{\partial u}\cdot\frac{\partial^2 \mathbf{r}}{\partial u^2}\right)^2\right] + \sigma\right\}\sqrt{g}. \tag{7}$$

By applying Hamilton's principle, including a term for an external force $\mathbf{F}_{\text{ext}}$ per element du,

$$\delta\mathcal{L} = \delta\mathcal{L}_{id} + \int\int \mathbf{F}_{\text{ext}} \cdot \delta\mathbf{r} du dt = 0, \tag{8}$$

one can express the force exerted by the interface upon the fluids, $\mathbf{F} = -\mathbf{F}_{\text{ext}}$, as a variational derivative

$$\mathbf{F} = \frac{\delta\mathcal{L}_{id}}{\delta\mathbf{r}}. \tag{9}$$

As in Ref. [1], the boundary terms vanish for closed surfaces or periodic boundary conditions, and the variational derivative is

$$\frac{\delta\mathcal{L}_{id}}{\delta\mathbf{r}} = -\frac{\partial}{\partial t}\frac{\partial L_{id}}{\partial\dot{\mathbf{r}}} - \frac{\partial}{\partial u}\frac{\partial L_{id}}{\partial\mathbf{r}'} + \frac{\partial^2}{\partial u^2}\frac{\partial L_{id}}{\partial\mathbf{r}''}. \tag{10}$$

Substituting (6) into (10) and then into (9), taking into account (3) and setting $g=1$ and all the derivatives of g to zero *after* the differentiation, we obtain

$$\mathbf{F} + \rho\frac{\partial^2 \mathbf{r}}{\partial t^2} = \left[\epsilon\left(H'' + \frac{1}{2}H^3\right) - \sigma H\right]\mathbf{n}, \tag{11}$$

where H'' is the second derivative of the local curvature with respect to the arclength in a canonical parametrization.

The formula (11) is different from the Eq. (2.3) given in Ref. [1] and may result in force different in magnitude and sign. The validation simulations undertaken in Ref. [1] were in agreement with the theoretical calculations because both were based on the same formula (2.3, Ref. [1]) that does not take into account the variation of metrics.

Applying an equivalent method to the Lagrangian (1) for the two-dimensional surface embedded in the three-dimensional space yields the expression

$$\mathbf{F} + \rho\ddot{\mathbf{r}} = [\epsilon(\nabla^2 H + \tfrac{1}{2}H^3 - 2HK) - \sigma H]\mathbf{n}, \tag{12}$$

where $H = \gamma_1 + \gamma_2$ is the mean curvature, $K = \gamma_1\gamma_2$ is the Gaussian curvature, γ_1 and γ_2 are the principal curvatures of the surface, $\nabla^2 = \partial^2/\partial u_1^2 + \partial^2/\partial u_2^2$, $\mathbf{r} = \mathbf{r}(u_1, u_2, t)$ is the canonical parametrization of the two-dimensional surface, satisfying the condition

$$\frac{\partial \mathbf{r}}{\partial u_i} \cdot \frac{\partial \mathbf{r}}{\partial u_j} = \delta_{ij}. \tag{13}$$

The equation (12) disagrees with the equation (A13) of Ref. [1] for the reasons considered above. An additional term in the free energy considered in Ref. [1],

$$\overline{\epsilon} \int K dS, \tag{14}$$

which is related to the Gaussian curvature and characterized by the saddle-splay modulus $\overline{\epsilon}$, gives no contribution to the force in accordance with the Gauss-Bonnet theorem which implies that the integral (14) does not change for the variations of the surface that do not change its topology, and hence the variation of (14) is zero. This is supported by the explicit calculations with the described method, but contradicts the equation (A13) obtained in the [1].

[1] D. Stelitano and D. H. Rothman, Phys. Rev. E **62**, 6667 (2000).